Excess continued fraction expansions

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***Abstract*— In the case of "simple" infinite continued fraction expansions of real numbers, the rational approximants have the property that if they are even, they increase with *n* increasing while if they are uneven, they decrease strictly with *n* increasing.**
Looking for a quicker convergence of the rational approximants, we have found a quadratic equation, which solutions can be expressed as a continued fraction expansion that we call "excess continued fraction expansion" because all its rational approximants are in excess.
Besides, we have proved that, if we consider the Golden Mean and the sequence of its successive powers, the uneven powers have a purely periodic continued fraction expansion and the even powers have an excess continued fraction expansion which converges much faster than a simple one.
For other members of the Metallic Means Family (introduced by the author in 1997), the behavior is similar for the sequence of the powers of the subfamily of Metallic Means which have a purely periodic continued fraction expansion.

***Keywords*— continued fraction expansions, excess continued fraction expansions, rational approximants, Metallic Means Family, Golden Mean**

I. Introduction

The continued fraction expansion

$$x = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}}$$

where a_ν, b_ν are integers, is called "half-regular" if

$$|a_\nu| = 1 \text{ for } \nu \geq 1$$
$$b_\nu \geq 1; b_\nu + a_{\nu+1} \geq 1 \text{ for } \nu \geq 1.$$

As is well known, for these half-regular continued fraction expansions, Tietze [1] has proved a convergence criterion and Perron [2] has established that if the expansion is finite, *x* is a rational number and if it is infinite, *x* is an irrational number. These expansions are unique.

If the conditions

$$b_\nu = b_{\nu+k}\ ;\ a_{\nu+1} = a_{\nu+k+1}$$

are satisfied for $\nu = 0,1,...; k = 1,2,...$, the half-regular continued fraction is called "purely periodic". If they are satisfied only from a certain ν, they are "periodic". It has also been proved [3] that a periodic half-regular continued fraction represents a positive quadratic irrational number and if we take only a finite number of terms like

$$b_0 + \frac{a_1}{b_1} = \sigma_0\ ;\ b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2}} = \sigma_1\ ;\ \ldots$$

we get a sequence of rational approximations σ_n to the number *x* that converges to *x* when $n \to \infty$.

Finally, if $a_\nu = 1$ for every ν, that is

$$x = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots}}$$

the continued fraction expansion is called "simple" and if $a_\nu = -1$ for every ν,

$$x = b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \ddots}}$$

it is called "reduced" and then, *x* is a quadratic irrational number and reciprocally, the expansion of a half-regular continued fraction expansion of a quadratic irrational number is periodic.

II. THE METALLIC MEANS FAMILY

It has been proved [4] that in the case of simple infinite continued fraction expansions, the even rational approximants σ_n increase strictly with n increasing while the uneven rational approximants decrease strictly with n increasing. That is, the value of a simple continued fraction expansion is greater than any even rational approximant and less than any uneven rational approximant.

The Metallic Means Family (MMF) was introduced by Vera W. de Spinadel at the national level in [5] and at the international level in [6]. Its most prominent member is the well known Golden Mean and among its relatives we can mention the Silver Mean, the Bronze Mean, the Copper Mean, the Nickel Mean, etc. These numbers may be obtained as the positive solutions of quadratic equations of the type

(1) $$x^2 - px - q = 0 \quad \text{where } p, q \in N.$$

If we consider the case where q = 1 we have:

(2) $$x^2 - px - 1 = 0 \quad \Rightarrow \quad x = \frac{p + \sqrt{p^2 + 4}}{2}$$

It has been proved [7] that the positive solutions of this set of quadratic equations are the irrational quadratic numbers which continued fraction expansion is

(3) $$x = [\overline{p}] = p + \cfrac{1}{p + \cfrac{1}{p + \ddots}}$$

where $p \in N$, that is, a “purely periodic” expansion. We shall denote this subfamily of the MMF as PPMMF.

For p = 1, we have the Golden Mean

(4) $$\phi = [\overline{1}] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}$$

which possesses numerous applications of aesthetic, mathematic, harmonic, etc., nature. Geometrically, it is associated to the pentagonal symmetry because it is the value of the diagonal of a regular pentagon of unitary side $\phi = 1{,}618\ldots$, see [7].

Then we have the Silver Mean

(5) $$\sigma_{Ag} = [\overline{2}] = 1 + \sqrt{2}$$

related to the octagonal symmetry, since it is the value of the second diagonal of a regular octagon of unitary side. We shall indicate it $\sigma_{Ag} = \theta$.

For p = 3 we have the Bronze Mean

(6) $$\sigma_{Br} = [\overline{3}] = \frac{3 + \sqrt{13}}{2}$$

etc.

Instead, if we look for positive solutions of quadratic equations of the type

(7) $$x^2 - x - q = 0$$

we find natural numbers and other members of the MMF, which continued fraction expansions are of the form

(8) $$x = [m, \overline{n_1, n_2, n_3, \ldots}]$$

that is, a periodic continued fraction expansion.

Obviously, if q = 1, the positive solution is ϕ. If q = 2 we have the Copper Mean

(9) $$\sigma_{Cu} = 2 = [2, \overline{0}].$$

If q = 3 we obtain the Nickel Mean

(10) $\sigma_{Ni} = [2,3,3,3,...] = [2,\bar{3}] = \frac{1+\sqrt{13}}{2}$

and so on.

From the point of view of the Chemistry and Physics of the solid state, there are only two types of materials, amorphous and crystalline.

The crystalline materials enjoy a great symmetry of order 2, 3, 4 and 6. But in 1984, the discovery of an Aluminum compound which diffraction schema had symmetry of order 5, introduced a third new type of materials, the so called “quasi-crystals”.

This is a new research field but in the observed cases we find as self-similarity ratios the values $\frac{1+\sqrt{5}}{2}$, $1+\sqrt{2}$ and $2+\sqrt{3}$.

The two first values are Metallic Means and the third one is a solution of the equation $x^2 - 4x + 1 = 0$. The solutions greater than 1 of quadratic equations of the type

(11) $x^2 - px + 1 = 0$, $p \in N$,

are not Metallic Means but deserve to be considered because they are related to some members of the PPMMF.

The positive solutions of this equation are of the form

(12) $x = \frac{p+\sqrt{p^2-4}}{2}$ p >2

III. EXCESS CONTINUED FRACTIONS

As is easy to verify, the even powers of the PPMMF are always a solution of (11), see [8]. In example

$$\phi^4 - 3\phi^2 + 1 = 0$$
$$\phi^8 - 7\phi^4 + 1 = 0$$
$$\theta^4 - 6\theta^2 + 1 = 0$$
$$\theta^8 - 34\theta^4 + 1 = 0$$
$$(\sigma_{Br})^4 - 11(\sigma_{Br})^2 + 1 = 0 \quad ,$$
$$(\sigma_{Br})^8 - 119(\sigma_{Br})^4 + 1 = 0 \quad , \quad \ldots$$

The solutions greater than 1 of the quadratic equation (11), that we shall indicate τ_p, have a continued fraction expansion, of period 2, of the form:

$$\tau_p = \frac{p+\sqrt{p^2-4}}{2} =$$

(13) $$(p-1) + \cfrac{1}{1+\cfrac{1}{(p-2)+\cfrac{1}{1+\cfrac{1}{(p-2)+\ddots}}}}$$

$$= [p-1;\overline{1, p-2}]$$

Calling $F = \tau_p - (p-1) > 0$ we obtain

$$F = \cfrac{1}{1+\cfrac{1}{(p-2)+F}} \Leftrightarrow F = \frac{(p-2)+F}{(p-1)+F} \Leftrightarrow .$$
$$F^2 + (p-2)F + (2-p) = 0 \ , \quad p = 3,4,5,\ldots$$

This last quadratic equation admits only a positive solution such as

$$F = \frac{(2-p)+\sqrt{(p-2)^2 - 4(2-p)}}{2} = \frac{(2-p)+\sqrt{p^2-4}}{2}$$

Consequently

$$\tau_p = \frac{p+\sqrt{p^2-4}}{2} = (p-1) + \frac{(2-p)+\sqrt{p^2-4}}{2} = (p-1) + F = (p-1) + \cfrac{1}{1+\cfrac{1}{(p-2)+F}}$$

.

For $n=3,$ equation (11) has the form

$$x^2 - 3x + 1 = 0,$$

from where, dividing by $x \neq 0$ we get

$$x = 3 - \frac{1}{x}.$$

Replacing iteratively the value of *x*, we obtain

$$x = 3 - \cfrac{1}{3 - \cfrac{1}{3 - - \ddots}} = \left[\overline{3-}\right]$$

a purely periodic continued fraction expansion for which we adopt the notation $\left[\overline{n-}\right]$.

We call them "excess continued fractions" because the rational approximants are all greater than the value of the irrational quadratic number. In fact

$$x = \frac{3+\sqrt{5}}{2} = 2{,}6180339...$$

$$\sigma_1 = 3; \sigma_2 = \frac{8}{3} = 2{,}66...; \sigma_3 = \frac{21}{8} = 2{,}625...;$$

$$\sigma_4 = \frac{55}{21} = 2{,}619...$$

It is easy to verify that the fourth rational approximant has two decimal figures exact.
Returning to the members of the PPMMF, if we look for the continued fraction expansion of the powers of these means, we find the following results for the Golden Mean ϕ

$$\begin{aligned}
\phi^2 &= \left[2,\overline{1,1}\right]\\
\phi^3 &= \left[\overline{4}\right]\\
\phi^4 &= \left[6,\overline{1,5}\right]\\
\phi^5 &= \left[\overline{11}\right]\\
\phi^6 &= \left[17,\overline{1,16}\right]\\
\phi^7 &= \left[\overline{29}\right]\\
\phi^8 &= \left[46,\overline{1,45}\right]\\
\phi^9 &= \left[\overline{76}\right]\\
\phi^{10} &= \left[122,\overline{1,121}\right]
\end{aligned}$$

....

Evidently, in the case of the even powers, the continued fraction is periodic, being the first coefficient equal to the sum of the two numbers forming the period, while in the case of uneven powers, the continued fraction is purely periodic.

III. THEOREM

Theorem: The uneven powers of the members of the subfamily PPMMF have a purely periodic continued fraction expansion while the even powers of the members of the subfamily PPMMF, have an "excess continued fraction expansion".

Proof: Let us consider the case of the most prominent member of the subfamily PPMMF, the Golden Mean ϕ. If we take the Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …

where

(14) $$F(n+1) = F(n) + F(n-1) \qquad n \geq 1$$

it is well known that

(15) $$\lim_{n\to\infty}\frac{F(n)}{F(n-1)}=\phi .$$

Starting with other couple of values and keeping the recursion law, it is possible to obtain another type of generalized Fibonacci sequence defined by

(16) $$L(n+1)=L(1)F(n)+L(0)F(n-1)$$

For $L(0)=1; L(1)=3$ we get the following sequence

1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, …

that is known as "Lucas sequence", introduced in 1877 by Edouard Lucas [9]. Obviously

$$\lim_{n\to\infty}\frac{L(n)}{L(n-1)}=\phi .$$

A closed form for the Lucas numbers is

(17) $$L(n)=\phi^n+\frac{(-1)^n}{\phi^n}.$$

From here, we can get an expression for the odd powers of the Golden Mean. In fact

$$\phi^{2n+1}=L(2n+1)+\frac{1}{\phi^{2n+1}}$$

$$=L(2n+1)+\cfrac{1}{L(2n+1)+\cfrac{1}{L(2n+1)+\ddots}}=\left[\overline{L(2n+1)}\right]$$

(18) $$\phi^{2n+1}=\left[\overline{L(2n+1)}\right] \qquad n=0,1,\ldots$$

a purely periodic continued fraction expansion.

For the even powers of the Golden Mean, we will consider that equation (17) can be rewritten in the form

$$(\phi^{2n})^2-L(n)\phi^{2n}+1=0$$

a quadratic equation of type (11). Putting $L_1(2n)=L(2n-2)$ we have

$$(\phi^{2n}-1)\phi^{2n}-(\phi^{2n}-1)=L_1(2n)\phi^{2n}$$

from where we get

$$\phi^{2n}-1=L_1(2n)+\cfrac{1}{1+\cfrac{1}{\phi^{2n}-1\ddots}}=\left[\overline{L_1(2n),1}\right].$$

Finally

$$\phi^{2n}=\left[L_1(2n)+1,\overline{L(1),L_1(2n)}\right]$$

that can be written

(19) $$\phi^{2n}=\left[L(2n)-1,\overline{L(1),L(2n-2)}\right] .$$

For other members of the PPMMF, it is easy to prove similar results, since all of them can be obtained as limits of ratios of two consecutive terms of "generalized secondary Fibonacci sequences" [7], which terms satisfy conditions of the type

(20) $$G(n+1)=pG(n)+G(n-1)$$
$$p\geq 1$$

If we assume $p=2$ in equation (20), we get the Silver Mean θ and if $p=3$ we have the Bronze Mean σ_{Br} and so on. In this way we have proved that the uneven powers of the members of the PPMMF are purely periodic, while the even powers have an excess continued fraction which converges much faster than the common continued fraction expansion.

Examples:

a)

$$(\phi^2)^2 - 3\phi^2 + 1 = 0 \Rightarrow x = 3 - \frac{1}{x} \Rightarrow x = \left[\overline{3-}\right]$$

b)

$$(\phi^4)^2 - 4\phi^4 + 1 = 0 \Rightarrow x = 4 - \frac{1}{x} \Rightarrow x = \left[\overline{4-}\right]$$

etc.

IV. CONCLUSIONS

The expressions of the excess continued fractions expansions we have introduced, are not unique, but obviously, in the case of the even powers of the members of the PPMMF, their rational approximants are the better ones.

It is interesting to mention that the powers of all the members of the PPMM enjoy similar properties as the Golden Mean ϕ, that is, the odd power continued fraction expansions are purely periodic while the even powers are periodic of period length 2, satisfying the relation that the first coefficient is equal to the sum of the next two. Indeed

$$\theta = \left[\overline{2}\right];\ \theta^2 = \left[5,\overline{1,4}\right];\ \theta^3 = \left[\overline{14}\right];\ \theta^4 = \left[33,\overline{1,32}\right];$$
$$\theta^5 = \left[\overline{82}\right];\ \theta^6 = \left[197,\overline{1,196}\right];\ \theta^7 = \left[\overline{478}\right];$$
$$\theta^8 = \left[1153,\overline{1,1152}\right];\ \theta^9 = \left[\overline{2786}\right];\cdots$$

Similarly for

$$\sigma_{Br} = \left[\overline{3}\right];\ \sigma_{Br}^{\ 2} = \left[10,\overline{1,9}\right];\ \sigma_{Br}^{\ 3} = \left[\overline{36}\right];$$
$$\sigma_{Br}^{\ 4} = \left[118,\overline{1,117}\right];\ \sigma_{Br}^{\ 5} = \left[\overline{393}\right];$$
$$\sigma_{Br}^{\ 6} = \left[1297,\overline{1,1296}\right];\cdots$$

q.q.d.

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